This second simulation study forces the spacecraft orientation to track a desired trajectory that is generated via the desired dynamics of Eqs. (6) and (7) utilizing $\bar{\omega}_d(t)$ as input. We select $\bar{\omega}_d(t)$ to be a soft start trajectory in the following manner:

$$\bar{\omega}_d(t) = \{0 \quad [1-\exp(-0.01t^2)]\sin(0.5t) \quad [1-\exp(-0.01t^2)]\cos(0.5t)\}^T \tag{57}$$

The initial conditions for the desired and actual attitude of the spacecraft were selected as

$$q_0(0) = q_{0d}(0) = \sqrt{0.1}$$
$$\boldsymbol{q}(0) = \boldsymbol{q}_d(0) = \begin{bmatrix}0 & \sqrt{0.45} & \sqrt{0.45}\end{bmatrix}^T \tag{58}$$

such that the unit quaternion constraint is satisfied. The initial value for $\omega_1(t)$ was chosen to be 0 rad^{-1}. We notice here that $\bar{\omega}_{d1}(t) = \omega_1(0) = 0$. The auxiliary signal $z_d(t)$ was initialized to be $[1.01 \quad 0]^T$. The control gains that resulted in the best tracking performance are

$$k_p = 10.0 \qquad k_a = 0.5 \qquad \gamma_0 = 1.0 \qquad \gamma_1 = 0.1$$
$$\varepsilon_1 = 0.01 \qquad k_1 = 20.0 \qquad k_2 = 20.0 \tag{59}$$

The torques were saturated to remain ± 10 N $\cdot$ m. From Fig. 5, we can see that the spacecraft orientation tracks the desired trajectory to a UUB neighborhood. From Fig. 6, the torques remain bounded for all times. Note that the UUB neighborhood can always be reduced by selecting ε_1 to be a smaller value.

Conclusions

In this Note, we have presented a nonlinear controller for the attitude tracking problem for a rigid underactuated spacecraft. For the reduced-order problem, that is, the spacecraft dynamics are neglected, the controller achieved uniformly ultimately bounded tracking provided the initial tracking errors are selected sufficiently small. Simulation results for the controller demonstrated the efficacy of the proposed strategy in achieving tracking for the underactuated spacecraft.

Acknowledgments

This work is supported in part by National Science Foundation Grant DMI-9457967, Office of Naval Research Grant N000014-99-1-0589, a Department of Commerce Grant, and an Army Research Office Automotive Center Grant. The authors would like to thank the reviewers for their constructive suggestions and a careful review of the manuscript.

References

[1]Crouch, P., "Spacecraft Attitude Control and Stabilization: Applications of Geometric Control Theory to Rigid Body Models," *IEEE Transactions on Automatic Control*, Vol. 29, No. 4, 1984, pp. 321–331.

[2]Byrnes, C., and Isidori, A., "On the Attitude Stabilization of Rigid Spacecraft," *Automatica*, Vol. 27, No. 1, 1991, pp. 87–95.

[3]Brockett, R., "Asymptotic Stability and Feedback Stabilization," *Differential Geometric Control Theory*, edited by R. Brockett, R. Millman, and H. Sussmann, Birkhäuser Boston, Cambridge, MA, 1983, pp. 181–191.

[4]Morin, P., Samson, C., Pomet, J., and Jiang, Z., "Time-Varying Feedback Stabilization of the Attitude of a Rigid Spacecraft with Two Controls," *Systems and Controls Letters*, Vol. 25, No. 5, 1995, pp. 375–385.

[5]Coron, J., and Kerai, E., "Explicit Feedbacks Stabilizing the Attitude of a Rigid Spacecraft with Two Control Torques," *Automatica*, Vol. 32, No. 5, 1996, pp. 669–677.

[6]Morin, P., and Samson, C., "Time-Varying Exponential Stabilization of a Rigid Spacecraft with Two Control Torques," *IEEE Transactions on Automatic Control*, Vol. 42, No. 4, 1997, pp. 528–534.

[7]Tsiotras, P., and Luo, J., "Control of Underactuated Spacecraft with Bounded Inputs," *Automatica*, Vol. 36, No. 8, 2000, pp. 1153–1169.

[8]Dixon, W. E., Dawson, D. M., Zergeroglu, E., and Zhang, F., "Robust Tracking and Regulation Control for Mobile Robots," *International Journal of Robust and Nonlinear Control: Special Issue on Control of Underactuated Nonlinear Systems*, Vol. 10, No. 4, 2000, pp. 199–216.

[9]Hughes, P., *Spacecraft Attitude Dynamics*, Wiley, New York, 1986, pp. 17–31.

[10]Krstić, M., Kanellakopoulos, I., and Kokotovic, P., *Nonlinear and Adaptive Control Design*, Wiley, New York, 1995, pp. 19–86.

Method of Unsteady Aerodynamic Forces Approximation for Aeroservoelastic Interactions

Iulian Cotoi* and Ruxandra M. Botez†
Ecole de Technologie Supérieure,
Montréal, Quebec H3C 1K3, Canada

Introduction

THE adverse interactions occurring between the three main disciplines unsteady aerodynamics, aeroelasticity, and servocontrols are called aeroservoelastic interactions. These interactions can be described mathematically by a system of equations in a state-space form. This system requires a different representation for the unsteady aerodynamic forces from that for the classical flutter equation. The unsteady aerodynamic forces, in the case of the classical flutter equation or aeroelasticity, are calculated by the doublet lattice method (DLM) in the frequency domain for a set of reduced frequencies k and Mach numbers M. Because time-domain linear time-invariant ordinary differential equations (LTI ODE) are required for using modern control theory design, several approximations for the unsteady aerodynamic forces in the s domain have been developed. There are mainly three formulations to approximate the unsteady generalized forces by rational functions in the Laplace domain in the frequency domain[1–4]: least square (LS), modified matrix Padé, and minimum state (MS). The approximation yielding the smallest order time-domain LTI state-space model is the MS (Ref. 5) approximation method. The dimension of the time-domain LTI state-space model depends on the number of retained modes and the number of aerodynamic lags n_a. There is a tradeoff between the number of aerodynamic lags and the accuracy of the approximation. The higher the n_a, the better the approximation, but the order of the time-domain LTI state-space model is larger. The order of the LTI state-space model strongly affects the efficiency of subsequent analyses. In the present Note a new method for the determination of efficient state-space aeroservoelastic models is presented. This method combines results from the theory of Padé approximants and the theory of model reduction developed in the frame of control theory. Finally, a comparison between the new method and the MS method is presented. The error of our method is 12–40 times smaller than the error of the MS approximation method for the same number of augmented states n_a and depends on the choice of the model reduction method.

Aircraft Equations of Motion

The motion of an aircraft modeled as a flexible structure with no forcing terms is described by the following equations, written in the time-domain:

$$\tilde{\boldsymbol{M}}\ddot{\eta} + \tilde{\boldsymbol{C}}\dot{\eta} + \tilde{\boldsymbol{K}}\eta + q_{\text{dyn}}\boldsymbol{Q}(k)\eta = 0 \tag{1}$$

Here η is the generalized variable defined as $\boldsymbol{q} = \boldsymbol{\Phi}\eta$, where $\boldsymbol{q}$ is the displacement vector and $\boldsymbol{\Phi}$ is the matrix formed with the

Received 20 July 2001; revision received 1 March 2002; accepted for publication 7 March 2002.

*Postdoctoral Fellow.
†Professor, Automated Production Department. Member AIAA.

eigenvectors of the free-vibration problem $M\ddot{q} + Kq = 0$. Moreover $\tilde{M} = \Phi^T M \Phi$, $\tilde{C} = \Phi^T C \Phi$, $\tilde{K} = \Phi^T K \Phi$ and $Q(k) = \Phi^T A_e(k) \Phi$, where M, K, and C are the generalized mass, the elastic stiffness, and the damping matrices; $q_{\text{dyn}} = 0.5\rho V^2$ is the dynamic pressure, where ρ is the air density; V is the true airspeed; $k = \omega b / V$ is the reduced frequency; ω is the natural frequency; b is the wing semichord length; $A_e(k)$ is the aerodynamic influence coefficient matrix for a given Mach number M and a set of $k \in \{k_1, k_2, \ldots, k_p\}$ values.

Applying the Laplace transform to Eq. (1), we obtain

$$[\tilde{M}s^2 + \tilde{C}s + \tilde{K}]\eta(s) + q_{\text{dyn}} Q(s)\eta(s) = 0 \tag{2}$$

The approximation of the unsteady aerodynamic forces is a necessary prerequisite to the control analysis of the subsequent aeroelastic system. Because $Q(k)$ can only be tabulated for a finite set of reduced frequencies, at a fixed Mach number M, it must be interpolated in the s domain in order to obtain $Q(s)$. Assuming analyticity of Q over the region of interest, one could find the Laurent series for $Q(s) = \sum c_i s^i$. This approach is not desirable from a control point of view because it introduces derivatives of order greater than two. Because we expect the linear aeroelastic system to behave as a bounded-input bounded-output (BIBO) system, it would seem obvious to find a rational fraction approximation for $Q(s)$. The method giving the lowest aerodynamic dimension is the MS approximation[5]:

$$Q^{\text{MS}}(s) = A_0 + A_1 s + A_2 s^2 + D(sI - R)^{-1} E \tag{3}$$

In the last term of the preceding equation, the denominator coefficients are the same as the ones considered in the LS method, and the numerator are calculated as a coupled product of D and E matrices. The diagonal matrix of aerodynamic roots is $R = \text{diag}(b_1, b_p)$, where the b_i are the lags that are usually chosen to belong in the range between zero and the highest frequency of vibration.

For a given set of lags b_p, the MS numerator coefficient matrices D and E are determined using an iterative, nonlinear LS methods that minimizes an overall error function J_{MS} (Ref. 1).

The iterative technique assumes an initial D and calculates E in order to calculate a new D matrix using LS methods. This iterative technique[5] continues until coefficient matrices are found with a converged minimum error.

Rational Function Approximation Method

In this section we present a new method for approximating the unsteady aerodynamic forces. Let us denote by Q_l^{TAB} the matrix formed with the tabulated value of the element $Q_{ij}(ik_l)$, where i denotes the pure imaginary number and k_l belongs to the set of reduced frequencies for which the DLM method is available. We approximate each element of the unsteady aerodynamic influence matrix by a $[N_{ij} + 2, N_{ij}]$ Padé approximation, where N_{ij} is a natural number depending on the element to be approximated and chosen in such way that we have a good accuracy for the approximation. Denoting by s the Laplace variable, we have

$$\bar{Q}_{ij}(s) = \frac{P_{N+2}(s)}{R_N(s)} = \frac{a_0 s^{N+2} + a_1 s^{N+1} + \cdots + a_{N+2}}{b_0 s^N + b_1 s^{N-1} + \cdots + b_N} \tag{4}$$

where the indices ij for the coefficients of the polynomials P_{N+2} and R_N have been omitted. For details on Padé approximants, see Ref. 6.

Because the aeroelastic system should be a BIBO system, we will impose that the coefficients of the denominator satisfy the Routh–Hurwitz criterion. The coefficients of the Padé approximants are found by using a standard LS technique and by minimizing the following error function:

$$J_{ij} = \left(\sum_{p=1}^{l} \left| \left[Q_p^{\text{TAB}} \right]_{ij} - \bar{Q}_{ij}(ik_p) \right|^2 \right)^{\frac{1}{2}} \tag{5}$$

for each couple (i, j). Here $|\cdot|$ denotes the modulus of a complex number. Once the coefficients of P_{N+2} and R_N are determined, it is a simple matter to write the aerodynamic approximation $\bar{Q}$ as follows:

$$\bar{Q}(s) = A_0 + A_1 s + A_2 s^2 + Z(s)s \tag{6}$$

where $Z(s)$ is a proper rational matrix. Indeed, each element of $\bar{Q}_{ij}$ is the ratio of a polynomial of degree $N + 2$ and a polynomial of degree N. For an arbitrary fixed couple of indices (i, j), let us find $\alpha_{0,ij}, \alpha_{1,ij}, \alpha_{2,ij}$, and $c_{0,ij}, c_{1,ij}, \ldots, c_{N-1,ij}$ such that for all s we will write Eq. (6) [for each couple (i, j)], as follows:

$$Q_{ij}(s) = \frac{a_0 s^{N+2} + \cdots + a_{N+2}}{b_0 s^N + \cdots + b_N} = \alpha_0 s^2 + \alpha_1 s + \alpha_2 + \frac{c_0 s^{N-1} + \cdots + c_{N-1}}{b_0 s^N + b_1 s^{N-1} + \cdots + b_N} s$$

We have dropped the indices ij in order to simplify the notation. By identification we obtain the following linear system:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ b_2 & b_1 & 1 & 1 & 0 & \cdots & 0 \\ b_3 & b_2 & b_1 & 0 & 1 & \cdots & 0 \\ b_4 & b_3 & b_2 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & b_N & b_{N-1} & 0 & 0 & \cdots & 1 \\ 0 & 0 & b_N & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ c_0 \\ c_1 \\ \cdots \\ c_{N-2} \\ c_{N-1} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \cdots \\ a_{N+1} \\ a_{N+2} \end{pmatrix}$$

From the first, the second, and the last equation, we obtain $(\alpha_i)_{i=\overline{0,2}}$. By backsubstitution the values of $(c_i)_{i=\overline{0,N-1}}$ are found.

It is easy to see that the determinant of the preceding system is b_N, and b_N is different from zero because it equals the product of the roots of $R_N(s)$, which have strictly negative real parts.

For every couple (i, j), we define $[A_p]_{ij} = \alpha_{2-p,ij}$ with $p = \overline{0, 2}$ and

$$Z_{ij}(s) = \frac{c_{0,ij} s^{N-1} + c_{1,ij} s^{N-2} + \cdots + c_{N-1,ij}}{b_{0,ij} s^N + b_{1,ij} s^{N-1} + \cdots + b_{N,ij}} \tag{7}$$

With these new definitions we find $\bar{Q}(s)$ by Eq. (6).

$Z(s)$ is a strictly proper rational matrix and therefore can be viewed as the transfer function of a linear system. We wish to find a triple (A, B, C) such that we have $Z(s) \approx C(sI - A)^{-1}B$. This can be achieved either by constructing a minimal-order realization (denoted by Minreal) or by constructing a reduced model starting from a known realization (for example, the canonical or the modal realization). The first approach gives an equality sign, whereas the second approach can be viewed as an approximation in a chosen norm.

A model-order reduction can be described as follows: Starting with a full-order model $Z(s)$, find a lower-order model $\hat{Z}(s)$ such that Z and $\hat{Z}$ are close (where the closeness is to be defined). This can be achieved either by an additive model-order reduction or by a relative-multiplicative model-order reduction. The additive model-order reduction consists in finding $\hat{Z}$ such that $Z = \hat{Z} + \Delta_{\text{add}}$, where Δ_{add} is small in some norm.

The relative-multiplicative model-order reduction comes to finding $\hat{Z}$ such that $\hat{Z} = Z(I + \Delta_{\text{rel}})$ and $Z = \hat{Z}(I + \Delta_{\text{rel}})$, where I is the identity and Δ_{rel} is small in some norm. The two additive methods available in the MATLAB® Robust Toolbox used in this Note are as follows: 1) Schur balanced model-order reduction (Schur) and 2) optimal Hankel approximation (Ohkapp).

The relative-multiplicative method is the balanced stochastic truncation (Bst-Rem) with relative error model reduction. For details concerning this methods, see the MATLAB documentation and the references therein.

We can therefore proceed as follows:

1) By eliminating all of the unobservable and uncontrollable states of the linear system whose transfer function is given by $Z(s)$, we construct a minimal-order transfer equivalent realization, which gives a perfect match, that is, $Z(s) = C(sI - A)^{-1}B$.

2) Finding $\hat{Z}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ such that $\|\hat{Z}(s) - Z(s)\|_\infty \le \varepsilon$, where $\varepsilon > 0$ is a small tolerance and $\|\cdot\|_\infty$ is the H^∞ norm,

Table 1 J and n_a calculations by different reduction methods

Modes	Minreal		Schur		Ohkapp		Bst-Rem		Error
N	n_a	J_{MS}	n_a	J_{MS}	n_a	J_{MS}	n_a	J_{MS}	J
5	17	3.69	19	3.56	18	3.99	6	11.16	0.26
10	81	6.16	80	6.16	79	7.36	31	11.28	0.51
15	192	10.40	187	10.46	82	16.46	37	23.27	0.73
20	344	23.45	337	23.78	152	24.16	69	38.19	1.07

we construct an optimal approximation of $\boldsymbol{Z}(s)$. If we denote by $\hat{\boldsymbol{Q}}(s) = \boldsymbol{A}_0 + \boldsymbol{A}_1 s + \boldsymbol{A}_2 s^2 + \hat{\boldsymbol{Z}}(s)s$, then we have

$$\|\hat{\boldsymbol{Q}}(s) - \boldsymbol{Q}^{\mathrm{TAB}}\|_\infty \le \|\hat{\boldsymbol{Q}}(s) - \bar{\boldsymbol{Q}}(s)\|_\infty + \|\bar{\boldsymbol{Q}}(s) - \boldsymbol{Q}^{\mathrm{TAB}}\|_\infty$$

$$\le \|\bar{\boldsymbol{Q}}(s) - \boldsymbol{Q}^{\mathrm{TAB}}\|_\infty + \varepsilon$$

This shows that the additive model-order reduction $\boldsymbol{Z}(s)$ by $\hat{\boldsymbol{Z}}(s)$ only degrades the norm of the approximation by an order of ε. A similar inequality can be showed for the relative-multiplicative model-order reduction.

Numerical Results

A flexible aircraft with 5, 10, 15, and 20 vibration modes has been considered. The finite element model of the symmetric one-half of an aircraft is used to verify this new optimization theory of unsteady aerodynamic forces.[7] The DLM implemented in STARS[7] was used to obtain the tabulated unsteady aerodynamic matrices in the frequency domain, for a given Mach number $M = 0.8$ and a set of 14 reduced frequencies $k \in \{0.01, 0.1, 0.2, 0.303, 0.4, 0.5, 0.5882, 0.6250, 0.6667, 0.7143, 0.7692, 0.8333, 0.9091, 1.0000\}$. With these tabulated data each element of the aerodynamic matrix is approximated by a Padé approximant. The global error of the approximation of our method is reported in the last column of Table 1. Following the procedure developed in the preceding section, we construct a strictly proper rational matrix from the matrix of Padé approximants [Eq. (7)]. Considering the strictly proper rational matrix as the transfer function of a linear system, we construct a reduced-order model using different reduction methods from control theory.

Eliminating the unobservable and the uncontrollable states with the help of the Minreal function of MATLAB, we obtain a minimal realization of order n_a, as shown in the second column in Table 1. Following the results of the preceding section, we construct the two additive reduced-order models, using the Schur and the Optimal Hankel approximation method. The reduced-order model for the latter two methods are reported in columns 4 and 6 of Table 1, respectively. The eighth column represents the dimension of the reduced system obtained using the balanced stochastic truncation method. We can see that this last method gives the best results in terms of aerodynamic dimension n_a in comparison to the other methods. The tolerance used in the calculation of the minimal realization and the approximation of the reduced-order models was chosen to be 10^{-6}. The aerodynamic dimensions n_a found by the four methods are now used to perform the MS approximation of the unsteady aerodynamic forces. This is done in order to compare the errors of unsteady aerodynamic forces approximation for a fixed aerodynamic dimension n_a. The errors for the MS method, denoted by J_{MS} for each method, are reported in columns 3, 5, 7, and 9 of Table 1. It can be seen that the approximation error calculated by our method is 12–40 times smaller than the errors calculated by the MS procedure.

Conclusions

The main contribution of this Note is an original method for the approximation of the unsteady aerodynamic forces using recent results from linear system theory and Padé approximants. Starting with a Padé approximation of the unsteady aerodynamic forces, we construct a reduced-order model for stability analysis purposes. Contrary to the standard MS approximation, the error of the approximation is independent on the aerodynamic dimension of the final aeroelastic system. Running the MS procedure with different numbers of lags to get a good approximation can take large amounts of time because the procedure is highly iterative. Our method overcomes the problem of choosing the number of lags (aerodynamic dimension) n_a of the MS procedure because in our method the aerodynamic dimension is a result not an initial parameter. Furthermore, our method yields better approximation error.

Acknowledgments

We would like to thank Vivek Mukhopadhyay for his comments and remarks on this Note. The authors would also like to thank Kajal Gupta at NASA Dryden Flight Research Center for allowing us to use STARS program and the Aircraft Test Model ATM. Many thanks are owed to the other members of the STARS Engineering group for their continuous assistance and collaboration: Tim Doyle, Can Bach, and Shun Lung.

References

[1] Tiffany, S. H., and Adams, W. M., "Nonlinear Programming Extensions to Rational Function Approximation of Unsteady Aerodynamics," NASA TP-2776, July 1988.

[2] Edwards, J. W., "Unsteady Aerodynamic Modeling and Active Aeroelastic Control," Stanford Univ., SUDAAR 504, Stanford, CA, Feb. 1977.

[3] Roger, K. L., "Airplane Math Modeling Methods for Active Control Design," *Structural Aspects of Active Controls*, CP-228, AGARD, Aug. 1977, pp. 4-1–4-11.

[4] Vepa, R., "Finite State Modeling of Aeroelastic System," NASA CR-2779, Feb. 1977.

[5] Karpel, M., "Design for Flutter Suppression and Gust Alleviation Using State Space Modeling," *Journal of Aircraft*, Vol. 19, No. 3, 1982, pp. 221–227.

[6] Baker, G. A., *Padé Approximants*, Cambridge Univ. Press, Cambridge, England, U.K., 1996, pp. 1–746.

[7] Gupta, K. K., "STARS-An Integrated Multidisciplinary, Finite Element, Structural, Fluids, Aeroelastic and Aeroservoelastic Analysis Computer Program," NASA TM-4795, 1997, pp. 1–285.

Composite Optimization Scheme for Time-Optimal Control

Michael C. Reynolds* and Peter H. Meckl†
Purdue University, West Lafayette, Indiana 47907-1288

I. Introduction

ALTHOUGH there are many techniques used to solve optimization problems, most of them can be categorized as constrained optimization methods. Constrained optimization involves the development and minimization of a cost function subject to a set of weighted constraints. These techniques are popular because they are usually easy to set up and they can be solved with many standard computational packages. But without a quality initial guess for the optimization routine, there might be no convergence or convergence to an undesirable solution. Finding a quality initial guess can be a difficult task in large systems of equations. A more systematic approach to finding an initial guess can be found by using linear programming. Linear programming can find the optimal solution to a system of equations without an initial guess. Starting with either linear or nonlinear equations, the problem is converted into a linear discrete form whose solution approximates the solution of the original problem. As the discrete intervals become smaller, the result approaches the continuous solution. The largest drawback of the linear programming technique is the lengthy computation time.

Received 31 July 2001; revision received 14 March 2002; accepted for publication 13 April 2002.

*Doctoral Candidate, School of Mechanical Engineering; reynoldm@purdue.edu.

†Associate Professor, School of Mechanical Engineering; meckl@ecn.purdue.edu.